

Special lagrangian fibrations on flag variety F^3

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Introduction

Lagrangian fibrations of symplectic manifolds are important for several applications.

From the classical mechanics point of view if a phase space is fibered on lagrangian submanifolds then the motion of the corresponding system with a Hamiltonian function whose hamiltonian vector field is parallel to the fibration can be described in terms of the fibers, and this can be done even in the case when certain fibers are singular.

An approach adopted in Geometric Quantization is realized in the same setup — if a given phase space is fibered on lagrangian submanifolds then one can take the so - called Bohr - Sommerfeld fibers which form a discrete subset, and this subset generates the corresponding Hilbert space. Here one should mention that the notion of Bohr - Sommerfeld lagrangian submanifold is valid for certain singular lagrangian fibers.

Homological Mirror Symmetry today can be treated by considerations of a lagrangian fibration and derivation a finite set of fibers which satisfy certain condition in terms of Fukaya - Floer cohomology. Again, one could try to count holomorphic discs for certain singular tori, so it seems that this approach to HMS can be extended to the cases when the fibrations have singular fibers.

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The ideal and model situation for the story originates in toric geometry, [1]. If one works with a symplectic toric manifold which is the phase space of a completely integrable mechanical system then there is a canonical lagrangian fibration of the manifold parameterized by a convex polytop and this leads to the introduction of the action - angle variables with complete solution of the system. At the same time in the toric case the methods for GQ and HMS mentioned above work succesfully. But the problem is that the set of toric manifolds is not too big and there are many other manifolds which do not admit a toric structure due to topological reasons but which must be involved in the consideration.

For a compact symplectic toric manifold (M, ω) the canonical lagrangian fibration is not completely regular — a dimensional reduction takes place so smooth lagrangian tori fill not whole M but the complement $M \setminus D$ where D is a "symplectic divisor" whose homology class is Poincare dual to the anticanonical class of M . This symplectic submanifold D of real codimension 2 consists of a number of irreducible smooth components D_i , and each component is as a symplectic manifold fibered on lagrangian tori which are isotropical being considered in M . Totally it can be described in terms of the moment polytop P : each facet of it is a moment polytop of a smooth symplectic toric manifold of real dimension equals to twice the dimension of the facet. In this setup it is reasonable to call D *the boundary divisor*.

Let us extend the class of symplectic toric manifolds. Here and in other papers, [3], [4], we study compact smooth symplectic manifolds which admit lagrangian fibrations of almost the same type as in the toric case unless the allowance of the existence of singular tori in the fibrations. We require that generic fiber must be smooth, and the types of the allowed singularties are exhausted by the following models. If T^n is a smooth n - dimensional torus, then we can take a decomposition

$$T^n = T^k \times T^{n-k}$$

and then the following precEDURE can be applied to our T^n : at a point $p \in T^{n-k}$ the fiber $T_p^k = T^k \times \{p\}$ collapses to $k - 1$ - dimensional torus. If we denote the corresponding degeneration operator as C_p^k , then we say that a singular torus which is allowed in our fibrations must be of the type $(C_{p_1}^{k_1} \dots C_{p_l}^{k_l}(T^n))$. Note that this type of singularities is familiar in classical mechanics — these correspond to separatrices; while a smooth torus in the fibration carries loop type trajectories, a singular torus can have lines and stable points as solutions.

In the present text we construct lagrangian fibrations of the type described above for complex flag variety F^3 , full flag variety for \mathbb{C}^3 . A lagrangian fibration of F^3 with boundary divisor D formed by 4 del Pezzo surfaces \mathbb{CP}_1^2 and with a subfamily of singular tori which has real dimension one is described below. Generic fiber of the fibration is smooth lagrangian torus, and the singular fiber is given by C_p^2 applied to T^3 . This fibration is not unique given by our construction, but we distinguish it and call it *minimal* since it has minimal set of singular tori. The construction is based on the fact that F^3 admits a *pseudo toric structure*, a generalization of the toric structure, proposed in [3] and [4]. Any pseudo toric manifold admits a variety of lagrangian fibrations of the desired type.

It is well known Gelfand - Zeytlin system on F^n , a completely integrable system, which is not however *effective* since F^n is not toric. The image of F^n under the "action" map is the Gelfand - Zeytlin polytop, and the difference between the toric case is manifested by the fact that the dimensional reduction is broken in this case: certain facets of the Gelfand - Zeytlin polytop are not the images of complex subsets of F^n , and they are not the images of certain submanifolds of right dimensions in F^n (of the dimensions equal to twice the dimensions of the facets), f.e. there are vertices whose preimages are not points in F^n (there is a vertex of maximal degeneration whose preimage is a lagrangian n - dimensional sphere). Thus our construction in contrast to the Gelfand - Zeytlin picture gives an alternative way to get a lagrangian fibration on F^n — we keep the dimensional reduction argument as in the toric case but we loose the smoothness of the fibers. Indeed, the boundary divisor for the Gelfand - Zeytlin system on the flag variety F^3 consists of 6 components: 2 complex and 4 non complex, and there is unique 4 - valent vertex (which lies in the intersection of these 4 components) in the Gelfand - Zeytlin polytop whose preimage is a lagrangian sphere in F^3 . In constrast the boundary divisor of our minimal lagrangian fibration consists of 4 complex components.

Why the minimal lagrangian fibrations of F^3 are interesting for us?

First, it gives new special solution to the classical mechanical problems on F^3 . F.e., one notes that three functions λ_j^i of the Gelfand - Zeytlin system can be combined to give two functions F_1, F_2 we are working with in our construction. This means that if one takes a Hamiltonian H on F^3 of the form

$$H = \alpha F_1 + \beta F_2$$

where $\alpha, \beta \in \mathbb{R}$, then the trajectories of the motion generated by H , can be found in two ways: either one takes the fibers of the Gelfand - Zeytlin lagrangian fibration and gets the motion restricting it to the fibers, or one can take the minimal fibration and study the fibers with the restricted motion. The answer, of course, must be the same, but perhaps in certain circumstances it would be convenient to work with our minimal fibrations, for example, in the problem of minimal loop searching.

Second, one can apply the standard methods of Geometric Quantization to F^3 endowed with the minimal lagrangian fibration. For a singular torus which appears in the fibration the notion of Bohr - Sommerfeld lagrangian cycle is still valid; the collapsed torus has a set of "virtual" periods, corresponding to the shrunk generators of the fundamental group; and these virtual periods must be trivial with respect to any integer symplectic form. Therefore one can define the set of Bohr - Sommerfeld fibers (with respect to the anticanonical class since F^3 is a Fano variety); one can prove that the set is discrete and finite, and thus the corresponding Hilbert space as usual is spanned by this finite set. The advantage of the minimal lagrangian fibration is a relationship between lagrangian and Kahler quantization for F^3 — as in the original toric case one has a version of the Danilov - Khovanskiy correspondence between holomorphic sections of the anticanonical class and the lagrangian cycle Bohr - Sommerfeld with respect to the anticanonical class. However for this set of problems we already have a classical result [5] which solves the Geometric Quantization problem in terms of representation theory.

Third, if one is interested in *special lagrangian fibrations* on Fano varieties, introduced recently by D. Auroux in [2], it is clear that if one studies the flag variety F^3 (or more generally F^n) then the Gelfand - Zeytlin fibration is not of the special type, and moreover it doesn't lead to something close to the desired type. At the same time the minimal lagrangian fibration on F^3 constructed below is special lagrangian, and this is the point of our main interest.

The last item is rather specific, but nevertheless it may lead to certain consistent results in the studies of lagrangian geometry of Fano varieties.

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1 Pseudotoric structure on the flag variety F^3

We construct a big family of lagrangian fibrations on F^3 using the notion of pseudotoric structures introduced in [3], [4]. We have the following

Proposition 1.1 *The flag variety F^3 is pseudotoric.*

Recall that pseudotoric structure on a symplectic manifold (M, ω_M) of real dimension $2n$ is a set of data $(f_1, \dots, f_k, B, \psi, (Y, \omega_Y))$ where f_i are smooth Morse functions in involution, ψ defines a family of symplectic toric submanifolds with the base set $B \subset M$ parameterized by a symplectic toric manifold (Y, ω_Y) of real dimension $2(n - k)$ so

$$\psi : M - B \rightarrow Y$$

is a map with generically smooth symplectic fibers, such that the following compatibility conditions hold:

- (1) for each $p \in Y$ the corresponding element of the family

$$\overline{\psi^{-1}(p)} = \psi^{-1}(p) \cup B$$

is preserved by the Hamiltonian action of each f_i ;

- (2) for each smooth function $h \in C^\infty(Y, \mathbb{R})$ we have the following identity

$$\nabla_\psi X_h = \tau X_{\psi^*h}$$

on $M - (B \cup \text{Sing})$, where Sing is the set of singularities of singular fibers of ψ , ∇_ψ is the symplectic connection defined by ψ , X_h and X_{ψ^*h} are Hamiltonian vector fields defined by ω_Y and ω_M respectively, and τ is a positive function.

In other words, if there is a toric action of T^k on M we are looking for the leaves of this action, a family of (generically smooth) symplectic toric manifolds with the same moment maps, parameterized by a symplectic toric manifold. Note that the compatibility condition (2) is trivial if both $(M, \omega_M), (Y, \omega_Y)$ are Kahler, ψ is a complex map and the complex dimension of Y is one.

As it is proven in [4], if a pseudotoric structure $(f_1, \dots, f_k, B, \psi, (Y, \omega_Y))$ is fixed on a symplectic manifold (M, ω_M) then the choice of moment maps (h_1, \dots, h_{n-k}) on Y defines a lagrangian fibration on M .

Now we are ready to prove Proposition 1.1. Consider the direct product $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$, fix homogenous coordinates $[x_0 : x_1 : x_2]$ and $[y_0 : y_1 : y_2]$ on the projective planes and realize F^3 as a hypersurface in $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ defined by the equation

$$\sum_{i=0}^2 x_i y_i = 0.$$

Consider the following map:

$$\psi : \mathbb{CP}_x^2 \times \mathbb{CP}_y^2 - B \rightarrow \mathbb{CP}_w^2,$$

defined by the equations

$$w_i = x_i y_i, i = 0, \dots, 2,$$

where $[w_0 : w_1 : w_2]$ are homogenous coordinates on the last \mathbb{CP}_w^2 . Since the coordinates are fixed, a standard Fubini - Study metric is fixed as well, and we endow the last \mathbb{CP}_w^2 by the corresponding symplectic form. The base set B consists of 6 lines defined either by

$$x_i = x_j = y_k = 0$$

or by

$$x_i = y_j = y_k = 0,$$

where i, j, k are distinct numbers from $\{0, 1, 2\}$.

What are the fibers of ψ ? It's not hard to see that for each $p \in \mathbb{CP}_w^2$ with coordinates $[w_0 : w_1 : w_2]$ such that $w_i \neq 0, i = 0, 1, 2$ the compactified fiber

$$\overline{\psi^{-1}(p)} = \psi^{-1}(p) \cup B$$

is a del Pezzo surface isomorphic to \mathbb{CP}_3^2 . The base set B is a complex six - angle on \mathbb{CP}_3^2 . This fiber is generic.

If now we take $p \in \mathbb{CP}_w^2$ with coordinates $[w_0 : w_1 : w_2]$ such that exactly one w_i equals to zero then the corresponding compactified fiber $\overline{\psi^{-1}(p)}$ is isomorphic to the union of two del Pezzo surfaces of the form \mathbb{CP}_1^2 which intersect each other at a "diagonal" of the six - angle B . Each \mathbb{CP}_1^2 has basic 4 - angle given by three edges of B and the intersection line. Thus the corresponding compactified fiber is singular with the singular set given by the "diagonal" line.

For a point $p \in \mathbb{CP}_w^2$ with coordinates $[w_0 : w_1 : w_2]$ such that only one w_i is non zero the compactified fiber is again non smoth — it consists of 2 \mathbb{CP}^2 and two quadrics. The six - angle B is divided into three pieces — two conjugated triangles and two lines which form a 4 - angles being taken with the diagonal lines. Triangles lie on \mathbb{CP}^2 s and 4 - angles on the quadrics.

Topologically each fiber is toric, and the construction endows each fiber with a divisor from the anticanonical system which can play the role of the degeneration simplex of the corresponding moment maps (or boundary divisor). Let us show that $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ carries a pair of moment maps whose Hamiltonian action preserves each fiber, so their restriction to each fiber gives a completely integrable system.

For each \mathbb{CP}_*^2 in the direct product we can consider a real function of the form

$$f(x) = \frac{\sum_{i=0}^2 \lambda_i^x |x_i|^2}{\sum_{i=0}^2 |x_i|^2},$$

where λ_i^x are distinct real numbers. In the same form one constructs a function $f(y)$ substituting y instead of x in the formula above. Then $f(x), f(y)$ are real Morse functions on the corresponding \mathbb{CP}_*^2 , whose Hamiltonian vector field preserves the Kahler structure. For two natural projections

$$q_* : \mathbb{CP}_x^2 \times \mathbb{CP}_y^2 \rightarrow \mathbb{CP}_*^2,$$

where $*$ is either x or y , a function of the form

$$F = q_x^* f(x) + q_y^* f(y)$$

(where now the star means the lifting) is a real Morse function on the direct product, whose Hamiltonian vector field preserves the Kahler structure on $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$, and it is not hard to see that if the following condition on λ_i^x, λ_j^y holds

$$\lambda_0^x + \lambda_0^y = \lambda_1^x + \lambda_1^y = \lambda_2^x + \lambda_2^y$$

then the Hamiltonian vector field of such a function F must preserve the fibers of ψ . Since we have on \mathbb{CP}^2 exactly two algerbaically independent almost everywhere real Morse functions of this form, it follows that there are exactly two algebraically independent almost everywhere Morse functions F_1, F_2 which preserve the fibers. The degeneration simplex

$$\Delta(F_1, F_2) = \{X_{F_1} \wedge X_{F_2} = 0\} \subset \mathbb{CP}_x^2 \times \mathbb{CP}_y^2$$

consists of the six - angle B and three its "diagonal" lines. Thus according to the general theory, see [4], the degeneration simplex is exactly the union of B and Sing .

Now we claim, that the constructed above data $(F_1, F_2, B, \psi, \mathbb{CP}^2)$ is a *pseudotoric structure* on the direct product $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$. To prove it remains to show that the second compatibility condition from the definition holds, which is not hard to do and we leave it aside the discussion. Our aim is to show that F^3 is pseudotoric as well.

The desired pseudotoric structure is given by the restriction of the data $(F_1, F_2, B, \psi, \mathbb{CP}^2)$ to the line $\mathbb{CP}_w^1 \subset \mathbb{CP}_w^2$ with the equation

$$w_0 + w_1 + w_2 = 0.$$

Then since for F_1, F_2 their Hamiltonian vector fields are tangent to F^3 and the image of F^3 under the map ψ is exactly the line \mathbb{CP}_w^1 we have the data $(f_1, f_2, B, \psi_1, \mathbb{CP}^1)$ where

$$\begin{aligned} f_i &= F_i|_{F^3}; \\ B &= B \text{ since } B \text{ lies on } F^3; \\ \psi_1 &= \psi|_{F^3-B}; \end{aligned}$$

and it's not hard to see that the data $(f_1, f_2, B, \psi_1, \mathbb{CP}_w^1)$ is a pseudotoric structure on F^3 .

The line $\mathbb{CP}_w^1 = \{w_0 + w_1 + w_2 = 0\} \subset \mathbb{CP}^2$ intersects three basic lines $l_i = \{w_i = 0\}, i = 0, 1, 2$, at three points, and these points underlay singular fibers of ψ . The type of the singular fibers was described above: each singular fiber is a pair of two \mathbb{CP}_1^2 (del Pezzo surfaces of degree 8) with the singular set represented by a "diagonal" line.

Now we are ready to present the minimal lagrangian fibration of F^3 , mentioned in the Introduction. To get it let us fix a smooth Morse function on \mathbb{CP}^1 which has two critical points, maximal and minimal, say, at $[0 : 1 : -1]$ and $[1 : 0 : -1]$. Then the boundary divisor $D \subset F^3$ of the corresponding fibration consists of two singular fibers, and each of these fiber is formed by two del Pezzo surfaces \mathbb{CP}_1^2 . Thus totally it gives 4 del Pezzo surfaces which are toric with the intergals F_1, F_2 , restricted to each component, and these integrals induce a canonical toric fibration of each component. It shows that D is fibered by smooth tori as usual in toric geometry. Now consider what happens on $F^3 - D$. Each fiber $\psi^{-1}(p)$ is fibered on smooth two dimensional tori by the restriction of f_1 and f_2 . These tori are lebeled by two regular values of f_1 and f_2 . Take a smooth level loop γ of the function h on \mathbb{CP}^1

which doesn't pass through the point $[1 : -1 : 0]$. Fix a pair of regular values for f_1, f_2 and collect the corresponding tori with the same values along γ :

$$\cup_{p \in \gamma} T_p^{c_1, c_2},$$

where $T_p^{c_1, c_2}$ is the torus from the fiber over p with the values c_1, c_2 . Then this union is a smooth three dimensional torus which is lagrangian in F^3 . This is a generic fiber.

Now when singular tori appear? If γ passes through the singular point $[1 : -1 : 0]$ it doesn't mean automatically that the corresponding torus must be singular. Indeed for the inner points of the six - angle in \mathbb{R}^2 which is the set of values for the pair f_1, f_2 a torus over $[1 : -1 : 0]$ is either 2 - dimensional or 1 - dimensional if the values (c_1, c_2) lie on the diagonal segment in the six - angle. Generic choice of (c_1, c_2) gives a 2 - dimensional torus, and collecting tori along γ we get a smooth lagrangian tori for this general case. The singularity appears if the pair (c_1, c_2) lies on the diagonal segment, and then this segment parameterizes singular tori of the following type: C_p^2 is applied to T^3 .

The lagrangian fibration of F^3 presented above is minimal in the following sense: another choice of a function h on \mathbb{CP}^1 underlying ψ_1 gives another fibration with bigger set of singular tori (except this h satisfies the same properties so its maximal and minimal points coincide with any two of the three marked points).

2 Specialty condition

Recall, that first time the notion of special lagrangian tori was presented for Calabi – Yau manifolds as a property, "dual" to the holomorphicity condition in the framework of the Mirror Symmetry problem. D. Auroux generalizes it in [2] for the case of certain Fano varieties. Namely, see [2], if a Fano variety X admits an effective divisor in the anticanonical system, say, $D \in |-K_X|$, then there exists a top holomorphic form θ_D on the complement $X - D$; this form has pole at D and doesn't vanish on $X - D$. Note that this form is uniquely defined by D up to \mathbb{C}^* .

Since by the very definition a Fano variety X can be embedded to a projective space by the anticanonical system one can attach to X the corresponding symplectic form given by the restriction to the image of X of

the standard symplectic form on the projective space. Thus any Fano variety carries a symplectic structure with the symplectic form, proportional to the (anti) canonical class, therefore any Fano variety can be treated as a monotone symplectic manifold; the symplectic structure on X can be call *canonical* (or anticanonical, if one wishes).

A lagrangian torus $S \subset X$ is *special* in (X, D) , if the imaginary part $\text{Im } e^{is}\theta_D|_S$ of the restriction $\theta_D|_S$ vanishes for certain twist $e^{is} \in U(1)$. A lagrangian fibration of $X - D$ is called *special* if the condition holds for any fiber with the same twist e^{is} .

The simplest example of special lagrangian fibration comes from the toric geometry consideration. Indeed, the canonical lagrangian fibration of a toric manifold X degenerates dimensionally at a divisor D_0 from the anticanonical system $|-K_X|$; then the corresponding holomorphic top form θ_{D_0} has very nice form in the action - angle coordinates which implies the property.

But the standard fibration is not quite good for several reasons, f.e. the boundary divisor D_0 is always reducible, and it was the problem stated in [2] to find a special lagrangian fibration for a smooth element from the anticanonical system on \mathbb{CP}^2 . The example of non standard special lagrangian fibration given in [2] has a reducible boundary divisor, the sum of line and conic, but it is more generic than the toric one.

At the same time the problem can be studied over any Fano variety with an effective divisor in the anticanonical system. F.e., we claim that the flag variety F^3 admits a special lagrangian fibration, and this fibration is our minimal one, constructed above.

Proposition 2.1 *The minimal lagrangian fibration on F^3 is special lagrangian.*

Indeed, the boundary divisor of the minimal fibration consists of two compactified fibers of ψ . The anticanonical class of F^3 as a bundle is given by the product $q_x^*\mathcal{O}_{\mathbb{CP}_x^2}(2) \otimes q_y^*\mathcal{O}_{\mathbb{CP}_y^2}(2)$ by the adjunction formula, and at the same time the compactified fibers of ψ are sections of the bundle $q_x^*\mathcal{O}_{\mathbb{CP}_x^2}(1) \otimes q_y^*\mathcal{O}_{\mathbb{CP}_y^2}(1)$ and thus the boundary divisor of the minimal fibration lies in the anticanonical system — the homological condition is satisfied.

Take now the boundary divisor $D \subset F^3$ of the minimal fibration and consider the corresponding top holomorphic form θ_D on $F^3 - D$.

We claim that there exist three real vector fields on $F^3 - D$, two global and one local, which preserves the holomorphic form θ_D . This means that the Lie derivative of θ_D along any one from this triple is zero, and since θ_D is closed

and the vector fields commute one gets a constant function substituting these fields to θ_D .

These three vector fields are: the Hamiltonian vector fields X_{f_1}, X_{f_2} (defined globally on whole F^3 and the lifted vector field $\nabla_\psi X_h$, which is defined outside of the singular points of ψ . Indeed, each X_{f_i} generates Hamiltonian isotopy of F^3 which leaves the boundary divisor D unmoved, and this implies that the Lie derivative $\mathcal{L}_{X_{f_i}} \theta_D$ is zero. On the other hand, the Hamiltonian vector field X_h on the base of the pseudotoric structure generates certain 1 - parameter family of symplectomorphisms of F^3 which moves fibers of ψ to fibers and which leaves unmoved the boundary divisor. This means that the Lie derivative $\mathcal{L}_{\nabla_\psi(X_h)} \theta_D$ vanishes (but now it is defined outside of the singular points of ψ only).

Consider now any torus $S \subset F^3$ from the minimal lagrangian fibration. Denote as $S_0 \subset S$ the "smooth" part consists of smooth points (so generically $S_0 = S$). At each point $p \in S_0$ the tangent space $T_p S$ is spanned by $X_{f_1}, X_{f_2}, \nabla_\psi X_h$. At the same time, since $\theta_D(X_{f_1}, X_{f_2}, \nabla_\psi X_h)$ is a constant complex function on S_0 and the determinant space $\det T_p S$ is real one dimensional, the restriction of θ_D to S_0 must be proportional to any real non vanishing top form with the proportionality coefficient of the form $f e^{i\phi}$ where f is a real function and ϕ is a constant. And the point is that this constant (so the argument) is the same for all $S \subset F^3$ from the minimal fibration. This ends the proof.

Here we implicitly use the fact that the function h is a symbol on the based \mathbb{CP}_w^1 , so its Hamiltonian vector field preserves the complex structure. Therefore the statement of Proposition 2.1 remains true for the lagrangian fibration on F^3 given by *any* symbol on the base \mathbb{CP}_w^1 .

3 Toric degeneration of F^3

We could describe the pseudotoric structure $(f_1, f_2, B, \psi_1, \mathbb{CP}^1)$ on the flag variety F^3 without any references to the representation by a hypersurface in the direct product $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$, but the computations above are usefull for the studying of toric degeneration of F^3 and a relationship between its canonical toric fibration and the minimal fibration of F^3 constructed above.

Toric degeneration of F^3 is a deformation of the flag variety to a singular toric variety. In the coordinates $\{x_i, y_i\}$ it is given by the following family

$$F_t = \{tx_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{CP}_x^2 \times \mathbb{CP}_y^2,$$

where t is a complex parameter. When $t \neq 0$ the variety F_t is isomorphic to F^3 while $t = 0$ implies that F_0 is a singular toric variety. As a toric variety F_0 carries a canonical lagrangian fibration. Our aim for this section is to relate the fibers of the lagrangian fibration of F^3 constructed in the previous section and the fibers of this canonical lagrangian fibration.

The first observation which is usefull in this way is the following

Proposition 3.1 *The singular toric variety F_0 carries a pseudotoric structure, and the standard toric fibration can be realized by the choice of an appropriate function h on the base \mathbb{CP}^1 .*

Indeed, consider again $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ and the pseudotoric structure $(F_1, F_2, B, \psi, \mathbb{CP}_w^2)$, presented in a previous section. Now we take the line

$$\mathbb{CP}_0^1 = \{w_1 + w_2 = 0\} \subset \mathbb{CP}_w^2;$$

it is clear that the Hamiltonian actions of F_1 and F_2 both preserve F_0 , therefore the restriction of F_i to F_0 gives a set of commuting functions, B is the base set of the family, the restriction of ψ to F_0 gives a map to \mathbb{CP}_0^1 which we denote as ψ_0 , and totally it gives a pseudotoric structure on F_0 .

The singular fibers of ψ_0 are the fibers over two points: $[1 : 0 : 0]$ and $[0 : 1 : -1]$ in \mathbb{CP}_0^1 . Now one can restore a standard toric fibration of F_0 :

Proposition 3.2 *Taking function h on \mathbb{CP}_0^1 such that it has exactly two critical points, at $[1 : 0 : 0]$ and $[0 : 1 : -1]$ one gets a canonical toric fibration of F_0 .*

One can show it in two steps:

— first, it's clear that a standard toric fibration on F_0 can be constructed from a standard toric fibration on the ambient variety $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ which is invariant with respect to F_1 and F_2 ;

— second, one can show that a standard toric fibration of the product variety $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ is defined by certain choice of moment maps H_1, H_2 on \mathbb{CP}_w^2 and the pseudotoric structure $(F_1, F_2, B_0, \psi, \mathbb{CP}_w^2)$.

These H_1, H_2 must be diagonal in the coordinates $[w_0 : w_1 : w_2]$ moreover they can be choosen such that one of them, say, H_1 is constant on the line $l_0 = \{w_0 = 0\}$. Then the Hamiltonian vector field of H_1 must be parallel to the line \mathbb{CP}_0^1 , and the restriction of H_1 to \mathbb{CP}_0^1 would be our h .

The only essential part of the proof is to show that diagonal in the coordinates $[w_0 : w_1 : w_2]$ functions H_1 and H_2 give a standard toric fibration

of $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$, so we clarify here only this point. To see this note that for the choice H_1, H_2 diagonal the singular fibers of ψ lie over the degeneration simplex $\Delta(H_1, H_2)$ of functions H_1, H_2 which is the union of lines $l_i = \{w_i = 0\} \subset \mathbb{CP}_w^2$. This means that the boundary divisor D of the corresponding fibration on $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ consists of the components:

$$\overline{\psi^{-1}(l_i)}, i = 0, 1, 2,$$

and these are precisely six irreducible components of the form $q_x^*(l_i^x), q_y^*(l_i^y), i = 0, 1, 2$, where $l_i^* \subset \mathbb{CP}_*^2$ is the projective line

$$l_i^x = \{x_i = 0\}, l_i^y = \{y_i = 0\}, i = 0, 1, 2.$$

Thus the boundary divisor for the fibration, generated by H_1, H_2 , is the same as for a standard toric fibration, and since for the diagonal functions H_1, H_2 the corresponding fibration has smooth fibers only, this fibration must be isomorphic to a standard fibration (given by the product of two Clifford fibrations on both $\mathbb{CP}_x^2, \mathbb{CP}_y^2$).

How one can compare lagrangian fibrations on F_0 and $F^3 = F_1$? The regular fibers can be related by certain Hamiltonian transformations of $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$. The procedure can be done as follows. Consider a Morse function g on \mathbb{CP}_w^2 such that its Hamiltonian vector field moves in a finite time the projective line $l_1 = \{w_0 + w_1 + w_2 = 0\}$ to the projective line $l_0 = \{w_1 + w_2\}$. If a Morse function h_1 is fixed on the projective line l_1 then it can be deformed to a Morse function h_0 on l_0 . At the same time one can define a Hamiltonian isotopy on $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ generated by a partial lift of g . Since ψ^*g is not defined at the base set $B \subset \mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ and ψ^*g is not smooth at the singular set Sing we take two small neighborhoods $B_1 \supset B_2$ of

$$\Delta(F_1, F_2) = B \cup \text{Sing} \subset \mathbb{CP}_x^2 \times \mathbb{CP}_y^2$$

of sufficiently small radii $1 \gg r_1 > r_2 > 0$; then one can construct a smooth function G on whole $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ by the following rules:

- $G = \psi^*g$ on $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2 - B_1$;
- G is zero in B_2 ;
- G smoothly changes from ∂B_1 to ∂B_2 .

Then it is clear that the Hamiltonian action of G on $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2$ moves a “central part” F_1^c of F_1 to a “central part” F_0^c of F_0 where one means the “central parts” to be

$$F_i^c = F_i - (F_i \cap B_1), i = 0, 1.$$

in a finite time. Therefore one can state the following

Proposition 3.3 *Every smooth lagrangian torus of the minimal lagrangian fibration of F^3 constructed in the previous section is Hamiltonically isotopic to a smooth lagrangian torus of the standard lagrangian fibration of the toric degeneration F_0 .*

To prove this fact one takes r_1 such that a given torus doesn't intersect B_1 ; it's easy to see that an appropriate choice exists. Then the Hamiltonian isotopy generated by G moves our given torus to a submanifold of F_0 which is fibered over the corresponding level loop of h_0 ; by the construction of G this function commutes with F_i everywhere outside of B_1 which implies that the resulting torus in F_0 lies on a mutual level set of the functions, and it follows that it is lagrangian in F_0 . It remains to see that it is given by the pseudotoric construction for the function h_0 which has exactly two critical points underlying singular fibers of ψ_0 , and hence it is a fiber of the standard fibration.

A singular torus via the limiting procedure $t \rightarrow 0$ collapses to an isotropical torus in F_0 which lies on a boundary component.

The last proposition implies the following fact:

Corollary 3.1 *The Floer cohomology $FH(S)$ of a smooth fiber $S \subset F^3$ in the minimal lagrangian fibration is the same as the Floer cohomology of a fiber of the standard lagrangian fibration of F_0 , the toric degeneration of F^3 .*

Indeed, the Floer cohomology (whatever it is defined) must be stable with respect to the Hamiltonian isotopy, and all the ingredients exploited for calculations of $FH(S)$ on F_0 can be moved to F^3 giving the same answer.

4 Digression: geometric construction

As we've mentioned above the pseudo toric structure on F^3 can be described in pure geometric terms. Here we discuss briefly how it can be done since probably it will be usefull for generic flag variety.

Consider \mathbb{C}^3 and fix a basis e_1, e_2, e_3 and a hermitian product \langle, \rangle such that $\langle e_i, e_i \rangle$ is an orthonormal basis for \langle, \rangle . Consider the projectivization \mathbb{CP}^2 of \mathbb{C}^3 ; the hermitian product defines a standard Kahler triple on \mathbb{CP}^2 , and the basis $\langle e_i \rangle$ fixes a "basic" triangle $\Delta \subset \mathbb{CP}^2$. Consider the flag

variety F^3 ; the hermitian product \langle, \rangle defines a Kahler triple on F^3 and the basis $\langle e_i \rangle$ gives six points on F^3 which correspond to ordered pairs $(e_i, e_j), i \neq j$; there are six edges which relate these six points such that one has a six - angle $\hat{\Delta} \subset F^3$.

Any self adjoint operator A on $(\mathbb{C}^3, \langle, \rangle)$ induces a pair of real functions defined up to constant on \mathbb{CP}^2 and F^3 . Namely A generates an infinitesimal symmetry of $(\mathbb{C}^3, \langle, \rangle)$ and thus the corresponding infinitesimal symmetries of \mathbb{CP}^2 and F^3 preserve the Kahler structures and therefore they are Hamiltonian and there are real functions f_A, F_A on \mathbb{CP}^2 and F^3 respectively defined up to constant. If one takes the standard projection

$$\pi : F^3 \rightarrow \mathbb{CP}^2$$

which is the forgetfull map for the second element in the flag, then it's not hard to see that π is a symplectic map so for each pair of tangent vectors $v_1, v_2 \in T_p F^3$ the value of the symplectic form $\omega_{F^3}(v_1, v_2)$ equals to $\omega_{\mathbb{CP}^2}(\pi(v_1), \pi(v_2))$ up to a multiple constant. It's not hard to see that the functions f_A and F_A are related by π as follows:

$$X_{f_A} = \pi(X_{F_A}).$$

Consider a pair of self adjoint operators A_1, A_2 which are diagonal with respect to the basis $\langle e_i \rangle$ with distinct eigenvalues, such that they are not proportional and their sum is not proportional to identity. Then they give two pair of Morse functions: f_1, f_2 on \mathbb{CP}^2 and F_1, F_2 on F^3 such that

$$X_{f_i} = d\pi(X_{F_i}).$$

The degeneration simplices

$$\Delta(f_1, f_2) = \{X_{f_1} \wedge X_{f_2} = 0\} \subset \mathbb{CP}^2$$

and

$$\Delta(F_1, F_2) = \{X_{F_1} \wedge X_{F_2} = 0\} \subset F^3$$

are related in a similiar manner

$$\pi(\Delta(F_1, F^3)) = \Delta(f_1, f_2),$$

and the first degeneration simplex $\Delta(F_1, F_2)$ consists of such flags $(p, l) | p \in l$ that the Hamiltonian actions of X_{f_1} and X_{f_2} on (p, l) are proportional.

Therefore it's clear that $\hat{\Delta}$ is a piece of $\Delta(F_1, F_2)$; it remains to add three "diagonal" lines in the six - angle $\hat{\Delta}$ to get $\Delta(F_1, F_2)$.

The choice of A_1, A_2 defines a singular connection ∇_A on the bundle $\pi : F^3 \rightarrow \mathbb{CP}^2$; the horizontal distribution over a point $p \in \mathbb{CP}^2$ which doesn't belong to Δ is given by the linear span of $X_{F_i}, I(X_{F_i})$ where I is the complex structure. This connection degenerates at $\pi^{-1}(\Delta)$. And a natural question arises — does this singular connection admits covariantly constant sections? Or is this distribution integrable? The integrability means the following: take a fiber of π and for each point of the fiber there is a (generically smooth) complex submanifold of F^3 which is

- projected to the base \mathbb{CP}^2 by the map π ;
- over each point except Δ the tangent space to this submanifold is horizontal with respect to the singular connection ∇_A .

Thus the section can meet each other over the degeneration set Δ and not outside.

Now we show that it is indeed the case:

Proposition 4.1 *The distribution, defined by the singular connection ∇_A , is intergable. Thus the connection ∇_A admits covariantly constant sections, and the space of the sections is parameterized by the fiber $\pi^{-1}(p)$ for a generic point $p \in \mathbb{CP}^2$.*

The proof is strightforward. We fix a point $p \in \mathbb{CP}^2$ outside of Δ and consider the fiber of π over p . The fiber is the pencil of lines passing through p . If we fix an element from the pencil, a line $l \subset \mathbb{CP}^2$, then there are two possibilities:

- either l doesn't intersect vertices of Δ , and this case is generic;
- or l passes through a vertex of Δ , and there are precisely three such elements in the pencil.

First we fix a generic line $l \ni p$ from the pencil and construct the corresponding covariantly constant section. For a fixed pair $(p, l) | p \in l$, consider the action of all non degenerated complex operators, diagonal in the basis $< e_i >$, up to scale, so the action of a complex 2 -torus. The orbit is isomorphic to $\mathbb{CP}^2 - \Delta$; denote it as $\psi^{-1}(p)$. Consider the degeneration simplex $\Delta(F_1, F_2)$ for the functions F_1, F_2 in F^3 . It consists of

- (i) 6 vertices which correspond to pairs (*vertex of Δ , edge of Δ*),
- (ii) six edges of two types, it (*vertex of Δ , a line through this vertex*) and (*point on edge of Δ , the edge of Δ*),

(iii) and three diagonal lines (*point on edge of Δ , line through vertex of Δ*) such that the point lies on the line.

The degeneration simplex $\Delta(F_1, F_2)$ is decomposed on two parts: the "six - angle" B is formed by (i) and (ii) and the subset Sing is (iii). The division is defined by the following argument: the six - angle B is mapped to a convex polytop in \mathbb{R}^2 by the action map

$$(F_1, F_2) : F^3 \rightarrow \mathbb{R}^2,$$

while the image of Sing under this map lies inside of the convex polytop. Now it is not hard to see that the union

$$\psi^{-1}(p) \cup B$$

is a smooth submanifold of F^3 birationally isomorphic to \mathbb{CP}^2 (since the projection π is a birational map). Moreover, the union

$$\psi^{-1}(p) \cup B$$

is isomorphic to del Pezzo surface \mathbb{CP}_3^2 . It is toric; moment maps are given by the restriction of F_1, F_2 to it. If one wants it can be established more carefully, step by step.

The first step: operators of rank 3. It's clear that the projective action of the diagonal operators of rank 3 on the flag (p, l) is transitive; each point of the base \mathbb{CP}^2 outside of Δ can be reached, and the second element of the pair $(K(p), K(l))$ is uniquely defined by the first one, where K is an operator. Thus one gets the part of orbit isomorphic to $\mathbb{CP}^2 - \Delta$.

The second step: operators of rank 2. If K has one zero eigenvalue then $(K(p), K(l))$ is of the following type: $K(p)$ lies on the corresponding edge of Δ and $K(l)$ is this edge. At the same the action of $K(p)$ is transitive, and this part of orbit is isomorphic to the union of three lines without the intersection points.

The third step: operators of rank 1. At this step the picture degenerates since l vanishes under the action of such an operator. At the same time point p goes to a vertex of Δ , so totally our orbit is projected to whole \mathbb{CP}^2 . But upstairs in F^3 the degeneration leads to the uncertainty at these vertices: the limiting procedure along edges gives different limiting lines at the vertex where the edges meet each other. This means that the appropriate smooth compactification must be given by the blow up of \mathbb{CP}^2 at the vertices of Δ .

And this is our covariantly constant section which is isomorphic to the del Pezzo surface \mathbb{CP}_3^2 embedded to F^3 horizontally. By the construction it is parallel to the Hamiltonian vector fields X_{F_i} almost everywhere.

It's clear that the horizontal section we've constructed starting with a general flag (p, l) contains the six - angle $B \subset F^3$ formed by edges and vertices of $\Delta \subset \mathbb{CP}^2$.

The section endowed with the restricted moment maps F_i gives a symplectic toric manifold, and the six - angle is precisely the boundary divisor. Note that each general horizontal section contains this basic six - angle by the construction.

Now consider the special case when the line l from the pair (p, l) passes through a vertex of Δ , say, the point $[1 : 0 : 0]$ in the homogenous coordinates $[z_0 : z_1 : z_2]$ on the base \mathbb{CP}^2 . Take again diagonal operators and study their action on (p, l) . Let's again study this case step by step and then put down certain generic arguments.

The first step: operators of rank 3. Again by the action of any such operator we can get any point outside of Δ in \mathbb{CP}^2 . At the same time all the lines we get will pass through the point $[1 : 0 : 0]$. It shows that this part is isomorphic to $\mathbb{CP}^2 - \Delta$.

The second step: operators of rank 2. Here we reach certain difference with the generic case since if K has the first eigenvalue equals to zero then the pair $(K(p), K(l))$ degenerates. Indeed, the line l should be transformed to the line $\{z_0 = 0\}$ but on the other hand it should pass through the point $[1 : 0 : 0]$ — and thus it vanishes. To complete this part one can restore the second element of the pair taking the line passing through $[1 : 0 : 0]$. Note that the line of F^3 which corresponds to the last case is a part of the degeneration simplex of f_1, f_2, f_3 on F^3 .

The third step: operators of rank 1. Near the point $[1 : 0 : 0]$ we have the same picture as in the general case so it must be blown - up. But now the other vertices are smoothly lifted to F^3 . Therefore this part of the compactified section is isomorphic to \mathbb{CP}_1^2 blown up at the point $[1 : 0 : 0]$.

At the same time one meets certain discrepancy: the constructed smooth section doesn't contain the basic six - angle B . Indeed, by the construction our section doesn't contain two vertices of B . At the same time it contains a diagonal line of B where the system of integrals f_1, f_2, f_3 degenerates. To recover the property that each element of the family contains a base set one needs

The fourth step: adding a component. Note that the component we get

above is a Schubert submanifold of F^3 . Indeed, it is presented by the condition that line l from flag (p, l) passes through a fixed point. Denote it by $D_{p_0} \subset F^3$, where p_0 is the point $[1 : 0 : 0]$. The diagonal line of the six - angle B at the same time is a Schubert submanifold as well: it consists of the flags (p, l) such that p lies on the line $\{z_0 = 0\}$ and l passes through the point $[1 : 0 : 0]$. The additional component one needs to take is "dual" to D_{p_0} : it is a Schubert submanifold consists of the flags (p, l) such that p lies on the line $l_0 = \{z_0 = 0\}$. We denote it as $D_{l_0} \subset F^3$. Now it's clear that $D_{x_0} \cup D_{l_0}$ is a singular submanifold, invariant under the Hamiltonian action of f_1, f_2, f_3 , which contains the base six - angle B . And it is our singular horizontal section, which corresponds to the element of the fiber $\pi^{-1}(p)$ passing through p_0 .

Since we have described all the possibilities for lines through p this ends the proof.

References

- [1] M. Audin, "Torus actions on symplectic manifolds", Second rev. ed., Progress in Math. 93, Birkhauser (2004).
- [2] D. Auroux "Mirror symmetry and T - duality in the complement of an anticanonical divisor", J. Gokova Geom. Top. 66: 1 (2007), pp. 51 - 91.
- [3] S. Belyov, N. Tyurin, "On non toric fibrations on lagrangian tori of toric Fano varieties", Math. Notes, 87, no. 1 (2010), pp. 48 - 59.
- [4] N. Tyurin, "Pseudotoric structure on toric and non toric Fano manifolds" Theor. Math. Phys. 162, no. 3 (2010), pp. 255 - 275.
- [5] V. Guillemin, S. Sternberg, "The Gel'fand - Cetlin system and quantization of the complex flag manifolds", J. Funct. Anal. 52 (1983), no. 1, 106-128.